CHAPTER

5

Structure and Logic

Algorithms and Algorithmic Thinking

Determine whether a given procedure for simplifying an expression is valid.

When you need to simplify an algebraic expression, there is an algorithm (a series of steps) called the Order of Operations that you must follow. Recall that, according to the Order of Operations, all operations contained within parentheses and other grouping symbols must be performed first. Absolute value signs and radicals are grouping symbols, so expressions within absolute value symbols or under radicals are evaluated at this time. After this, determine the values of powers and roots, and then perform multiplications and divisions in order from left to right. Conclude by performing additions and subtractions in order from left to right.

Any procedure that violates the Order of Operations is likely to lead to an incorrect answer and is therefore likely to not be valid. A procedure that does not violate the Order of Operations will be valid.

Example:

Below are three procedures for simplifying the expression $4 + 8 \div 2 \times 7 - 3$. Which one is valid?

Procedure A:

$$4 + 8 \div 2 \times 4 - 3$$
$$12 \div 2 \times 4 - 3$$
$$6 \times 4 - 3$$
$$24 - 3$$
$$21$$

Procedure B:

$$4 + 8 \div 2 \times 4 - 3$$
$$4 + 4 \times 4 - 3$$
$$4 + 16 - 3$$
$$20 - 3$$
$$17$$

Procedure C:

$$4 + 8 \div 2 \times 4 - 3$$
$$4 + 8 \div 8 - 3$$
$$4 + 1 - 3$$
$$5 - 3$$
$$2$$

Solution:
The Order of Operations requires that multiplications and divisions be performed before additions and subtractions and that multiplications and divisions should be evaluated in the order in which they occur, from left to right. ***Procedure A*** is not valid since the addition is performed first, prior to the multiplication and division. ***Procedure C*** is also not valid as the multiplication is performed before the division, even though the division is to the left of the multiplication. ***Procedure B*** however does follow the steps in the Order of Operations and is therefore valid.

Determine whether a given procedure for solving an equation is valid.

The procedure for solving a linear equation, essentially, involves successively rewriting the equation in equivalent, but simpler, forms, until the solution to the equation is obvious. These equivalent forms are obtained by performing the same operations on both sides of the equation. For example, if you were given the equation $8x + 6 = 4x + 2$ to solve, you might begin by subtracting 6 from both sides to obtain the equivalent equation $8x = 4x - 4$. As a next step, you might subtract $4x$ from both sides to obtain $4x = -4$. Finally, perhaps you would divide both sides by 4 to end up with $x = -1$.

This procedure, while perhaps the most straightforward way to solve the equation, is not the only series of steps that will lead to the correct solution. There are a variety of other procedures that can be performed, and as long as no rules of algebra are broken, these procedures can be equally valid and will lead to the same solution. For example, you might have started the procedure of solving the equation by dividing all terms by 2, to obtain the equation $4x + 3 = 2x + 1$. Next, you could have subtracted 1 from both sides to obtain $4x + 2 = 2x$. If you then subtract $4x$ from both sides, you will get $2 = -2x$. Finish by dividing both sides by 2, and again you will obtain $x = -1$. Since no algebraic mistakes were made and no rules were misapplied, the two procedures used, while different, are equally valid and result in the same solution.

Example 1:
Consider the equation $ax + b = c$. Below are four different procedures for solving this equation for x. Which ones are valid and which ones are not?

Procedure #1: Subtract b from both sides and then divide both sides by a.

Procedure #2: Divide all terms on both sides by a and then subtract $\frac{b}{a}$ from both sides.

Procedure #3: Divide both sides by $a + b$.

Procedure #4: Subtract c from both sides. Subtract ax from both sides. Then, divide both sides by $-a$.

Solution:
Procedure #1 is valid. When b is subtracted from both sides, you will obtain $ax = c - b$. Dividing both sides by a leads to the correct solution $x = \frac{c-b}{a}$.

Procedure #2 is also valid. If all terms are divided by a you will obtain $x + \frac{b}{a} = \frac{c}{a}$. Then, subtracting $\frac{b}{a}$ from both sides yields $x = \frac{c}{a} - \frac{b}{a}$. Note that this answer is mathematically equivalent to the answer $x = \frac{c-b}{a}$ obtained from ***Procedure #1*** above.

Procedure #3: This procedure is not valid. Note that, if you divide both sides by $a + b$, you will get $\frac{ax+b}{a+b} = \frac{c}{a+b}$. A careless algebraic error may lead you to conclude that you can cancel the $a + b$ from the top and bottom of the fraction on the left hand side, obtaining $x = \frac{c}{a+b}$ as the solution. However, this is not a correct cancellation, the answer is incorrect, and the procedure is not valid.

Procedure #4 is valid. If you subtract c from both sides, you get $ax + b - c = 0$. Subtracting ax from both sides leads to $b - c = -ax$. Then, dividing both sides by $-a$ gives $x = \frac{b-c}{-a}$, which is the same as $x = \frac{c-b}{a}$, which was obtained from ***Procedure #1.***

Determine whether a given procedure for solving a linear inequality is valid.

The procedure for solving a linear inequality is extremely similar to the procedure for solving a linear equation. Once again, you successively rewrite the inequality in equivalent, but simpler, forms until the solution is obvious. These equivalent forms are obtained by performing the same operations on both sides of the inequality. There are two primary differences between solving linear equations and solving linear inequalities. First of all, while a linear equation typically has one and only one solution, a linear inequality typically has an infinite number of solutions. And, most importantly from the standpoint of solving a linear inequality, whenever you multiply or divide by a negative number as a part of the